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The effect of viscosity and heat conduction on internal gravity waves at a critical level

By PHILIP HAZEL
Department of Applied Mathematics and Theoretical Physics, University of Cambridge

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The differential equation for the vertical velocity of a gravity wave in an inviscid shear flow is singular at a level where the mean fluid velocity is equal to the horizontal phase velocity of the waves. It has been shown that a wave travelling through such a layer has its amplitude attenuated by a constant factor dependent on the local Richardson number. In this paper the results obtained by solving numerically the full sixth order differential equation, which is derived by including viscosity and heat conduction in the problem, (and is not singular) are discussed, and *the same* attenuation factor is found. Some experiments which confirm certain aspects of the theory are described in an appendix.

Introduction

Recently Booker & Bretherton (1967) considered the problem of internal gravity waves in a shear flow with a critical level, i.e. a level where the mean fluid velocity is equal to the horizontal phase velocity of the waves. For an inviscid fluid, the differential equation for the vertical velocity is singular at such a level for all real phase velocities. However, by allowing the phase velocity to have a slightly imaginary part, and by looking at the asymptotics of the time-dependent initial value problem, they were able to obtain a matching condition across the singular point, namely that a wave travelling through the critical level has its amplitude attenuated by a factor of $e^{\mu\pi}$ where μ is given by $(R-\frac{1}{4})^{\frac{1}{2}}$, R being the Richardson number at that level. The linearized velocity obtained by this method eventually tends to infinity at the critical level, thus vitiating the linearization. Bretherton (1966) has also considered the wave-packet approach.

For a steady input, however, viscosity and heat conduction may play a more important role than non-linear effects. The differential equation for this problem is one of the sixth order, and is no longer singular at the critical level, even if the phase speed is kept strictly real. Two of its solutions tend to the inviscid solutions asymptotically *well away from* the critical level; the other four 'viscous' solutions are *not* negligible near this level, and so the coefficients of the inviscid solutions on either side must be determined from the full equation. It is not *a priori* evident that the matching condition across the critical level will be the same as that for the inviscid initial value problem, although this is known to be true for the critical level in a boundary layer. The equation in this case has a much worse singularity than the boundary-layer problem, *both* inviscid solutions being singular at the critical level. Nevertheless, numerical computations on the full sixth-order equation show that the matching condition is in fact the same as the one for the initial value problem.

1. The problem

We consider a two-dimensional, steady flow, with mean wind $(U(z), 0)$ and a perturbation $(u(x,z), w(x,z))$. Making a Boussinesq approximation, we then have the following linearized equations:

$$Uu_x + wU_z + (1/\rho)\, p_x - \nu\nabla^2 u = 0, \tag{1.1}$$

$$Uw_x + \sigma + (1/\rho)\, p_z - \nu\nabla^2 w = 0, \tag{1.2}$$

$$u_x + w_z = 0, \tag{1.3}$$

$$U\sigma_x - N^2 w = K\nabla^2\sigma, \tag{1.4}$$

where ν is the kinematic viscosity, K the coefficient of heat conductivity, N the Brunt-Väisälä frequency, and σ the buoyancy force per unit mass. Taking the Fourier transform of the vertical velocity, such that

$$w(x,z) = \int_{-\infty}^{\infty} \hat{w}(z,k)\, e^{ikx} dk,$$

elimination of u, σ and p leads to the general equation for $\hat{w}$, namely

$$[ikU - K(D^2-k^2)]\,[\{-ikU + \nu(D^2+2k^2)\}D^2 + ik(U_{zz}+Uk^2) + \nu k^4 + k^2N^2]\,\hat{w} = 0, \tag{1.5}$$

where $D = d/dz$.

If ν and K are sufficiently small, viscosity and heat conduction will only be important in a narrow region near the critical level. In this region we make the approximations that $k^2 \ll D^2$ (the hydrostatic approximation), and that N^2 and dU/dz are constant. Choose the origin of z such that $z = 0$ is the critical level. Then $U = U_z z$ in this region, and the equation simplifies to

$$\left[\left(\frac{iK}{kU_z} D^2 + z\right)\left(\frac{i\nu}{kU_z} D^2 + z\right) D^2 + R\right]\hat{w} = 0, \tag{1.6}$$

where R is the Richardson number N^2/U_z^2. A natural scale for the problem is given by $z_0 = (\nu/kU_z)^{\frac{1}{3}}$; the viscous terms being negligible for $|z| \gg z_0$. Thus we 'normalize' the equation by making the transformation $z = z_0\xi$, so that the inviscid region is now $|\xi| \gg 1$. The transformed equation is

$$\left[\left(iP^{-1}\frac{d^2}{d\xi^2} + \xi\right)\left(i\frac{d^2}{d\xi^2} + \xi\right)\frac{d^2}{d\xi^2} + R\right]\hat{w} = 0, \tag{1.7}$$

where P is the Prandtl number.

Limitations

Suppose the shear dU/dz is varying in the large over at length scale H. Then we must have $z_0 \ll H$ for the assumptions made in deriving (1.6) from (1.5) to hold.

Also we require $z_0 \ll 1/k$, which is satisfied for a wide range of k, including all practical cases.

Assuming the mean flow to be independent of x, the momentum equation for the mean horizontal velocity is

$$\frac{\partial U}{\partial t}+\frac{\partial}{\partial z}\overline{(uw)}+\frac{\partial p}{\partial x}-\nu U_{zz}=0, \tag{1.8}$$

where $\partial p/\partial x$ may depend on z. The second term represents momentum changes in the mean flow due to vertical changes in the Reynolds stress. Note that the mean flow is not in fact an exact solution of the equations;† however, the steady-state perturbation analysis can be justified by a comparison of time scales. The time scale for the variations in the mean flow due to viscosity is given by H^2/ν; we require that this be very much larger than the time scale for changes in the critical layer due to fluctuations in the input, i.e. the time scale for a steady state to develop. In their initial value analysis, Booker & Bretherton found the critical layer at any time to have thickness proportional to $(kU_z t)^{-1}$, hence the time scale for fluctuations in the critical layer is $(kU_z z_0)^{-1}$. Thus for the steady-state analysis to be valid, we require

$$\frac{1}{kU_z z_0}\frac{\nu}{H^2}=\left(\frac{z_0}{H}\right)^0 \ll 1.$$

This assumption has already been made above, and so steady-state analysis *is* justified.

2. Asymptotics for large $|\xi|$

Analytic solutions of (1.7) can be obtained in the form of power series or contour integrals. However, it proved impossible to match these to asymptotic series for large $|\xi|$ and so a numerical approach to the problem was used. Nevertheless, *some* information about the asymptotics is required to give starting values for the integration. A solution of the form $\hat{w}=\xi^m e^{q\xi^{\frac{3}{2}}}$ (m, q constant) is suggested by considering the equation in the case $P^{-1}=R=0$ for large $|\xi|$. From this it is straightforward, though tedious, to obtain the following leading terms for the six asymptotic solutions for large positive ξ and $(P \neq 1)$‡:

$$\hat{w}_1 = A_1\xi^{\frac{1}{2}+i\mu}\left(1+\frac{R(1+P^{-1})(2\mu+5i-[4i/1+P])}{12\xi^3}+O\xi^{-6}\right), \tag{2.1}$$

$$\hat{w}_2 = A_2\xi^{\frac{1}{2}-i\mu}\left(1+\frac{R(1+P^{-1})(2\mu-5i-[4i/1+P])}{12\xi^3}+O\xi^{-6}\right), \tag{2.2}$$

$$\hat{w}_3 = A_3\xi^{-\frac{5}{4}}e^{+\frac{2}{3}\sqrt{i}\,\xi^{\frac{3}{2}}}(1+O\xi^{-3}), \tag{2.3}$$

$$\hat{w}_4 = A_4\xi^{-\frac{5}{4}}e^{-\frac{2}{3}\sqrt{i}\,\xi^{\frac{3}{2}}}(1+O\xi^{-3}), \tag{2.4}$$

$$\hat{w}_5 = A_5\xi^{-\frac{9}{4}}e^{+\frac{2}{3}\sqrt{(iP^{\frac{1}{2}})}\,\xi^{\frac{3}{2}}}(1+O\xi^{-3}), \tag{2.5}$$

$$\hat{w}_6 = A_6\xi^{-\frac{9}{4}}e^{-\frac{2}{3}\sqrt{(iP)}\,\xi^{\frac{3}{2}}}(1+O\xi^{-3}), \tag{2.6}$$

where $\mu=(R-\frac{1}{4})^{\frac{1}{2}}$. Similar solutions, but with *different* constants, are obtained for large negative ξ. The leading terms of $\hat{w}_1$ and $\hat{w}_2$ are the solutions of the second-

† cf. stability theory.

‡ cf. Koppel (1964). The case $P=1$ has been given exhaustive analytical treatment by Koppel (1964) and Hughes & Reid (1967).

order inviscid equation, and represent an upward and a downward travelling wave respectively,† while the remaining 'viscous' solutions $\hat{w}_3$ to $\hat{w}_6$, are of exponential type, two decaying as $\xi \to +\infty$, and two as $\xi \to -\infty$.

3. Numerical solution

The problem posed for numerical solutions was to obtain a complete solution, continuous through the critical level, starting from an upward travelling wave of known amplitude at $-\infty$. The following procedure was adopted: as well as the input wave, a possible reflected wave and the two viscous solutions that go to zero at $-\infty$, all of unknown amplitude, were assumed to exist below the critical level, while above it a transmitted wave and the two other viscous solutions were assumed. Each of the four solutions in $\xi < 0$ and the three in $\xi > 0$ were integrated separately in to zero from equidistant points on their respective sides of the critical level. The unknown (complex) amplitudes were then determined by matching the solution and its first five derivatives at the origin. Using these amplitudes, a complete solution was obtained.

The starting values for the integrations were obtained from the asymptotic forms of §2, using the same conventions as Booker & Bretherton for the wave solutions, i.e. taking

$$\xi^{\frac{1}{2}+i\mu} = \begin{cases} |\xi|^{\frac{1}{2}} e^{i\mu \log|\xi|} & (\xi > 0), \\ -i|\xi|^{\frac{1}{2}} e^{\mu\pi} e^{i\mu\log|\xi|} & (\xi < 0), \end{cases} \tag{3.1}$$

and

$$\xi^{\frac{1}{2}-i\mu} = -ie^{-\mu\pi}|\xi|^{\frac{1}{2}} e^{-i\mu\log|\xi|} \quad (\xi < 0). \tag{3.2}$$

The other 'viscous' solutions were taken according to the same convention.‡

The best position for the starting-point of the integration was determined by trial and error. It was found that starting too near to zero gave a solution in which the viscous terms still played a significant part at the starting-point, while starting too far out introduced too much of the exponentially growing solutions. The value of $\hat{w}_5$, for example, at $\xi = 10$ is of the order of $10^{20} \times A_5$, and so any attempt to start further out is doomed to numerical failure. A starting-point in the region of $\xi = 6{\cdot}0$ was found to be the best compromise. A steplength of 0·05 was used most of the time, while the values of the Prandtl and Richardson numbers were taken as 7·0 and 3·0 respectively; the effect of varying all three of these, however, was investigated, although for $R > 4$, the numerical errors due to the large value of $e^{\mu\pi}$ gave inconsistent results, while for R close to $\frac{1}{4}$, the numerical problem became ill conditioned. The actual integration was carried out on the digital computer TITAN by means of a standard Runge–Kutta–Gill Library subroutine.

4. Results

(a) *Transmitted wave*

It was found, in agreement with Booker & Bretherton, that the amplitude of the wave transmitted through the critical level was severely attenuated by a factor close to their figure of $e^{\mu\pi}$. As this factor was explicitly included in the incident

† See Booker & Bretherton (1967) for a discussion of this interpretation.

‡ Note: in the published form of Booker & Bretherton's paper, there are misprints in these equations. The above forms are the correct ones.

Prandtl no.	Richardson no.	Steplength	Starting pt.	Coefficient: Real part	Coefficient: Imag. part	% difference
			4·6	1·0062	0·0042	0·1720
			5·2	1·0025	−0·0003	0·4800
		0·05	5·8	1·0003	−0·0007	0·6534
			6·2	0·9998	0·0008	0·6118
	3·0		6·8	1·0063	0·0102	−0·8209
		0·01		1·0011	0·0004	0·3633
7·0		0·005		1·0000	0·0000	0·5712
	0·2501			1·0002	0·0002	35·2079
	0·3			1·0000	0·0000	1·5987
	1·0			1·0000	−0·0001	1·1133
	2·0			0·9994	0·0000	1·0847
	4·0		6·2	1·0005	0·0005	−0·0983
1·1		0·05		1·0025	0·0004	0·4677
2·0				1·0001	−0·0003	0·7490
3·0	3·0			0·9999	−0·0002	0·6911
4·0				0·9999	−0·0001	0·6527
5·0				0·9999	−0·0002	0·6324
6·0				0·9997	−0·0004	0·6520

FIGURE 1. The column headed 'Coefficient' gives the complex coefficient of the transmitted wave, while the one headed '% difference' gives the percentage difference between the ratio of the Reynolds stresses on either side of the critical layer and $e^{2\mu\pi}$. A negative sign indicates that the ratio is less than $e^{2\mu\pi}$. The variations in the values of the coefficient are consistent with fluctuations due to truncation error in the starting values.

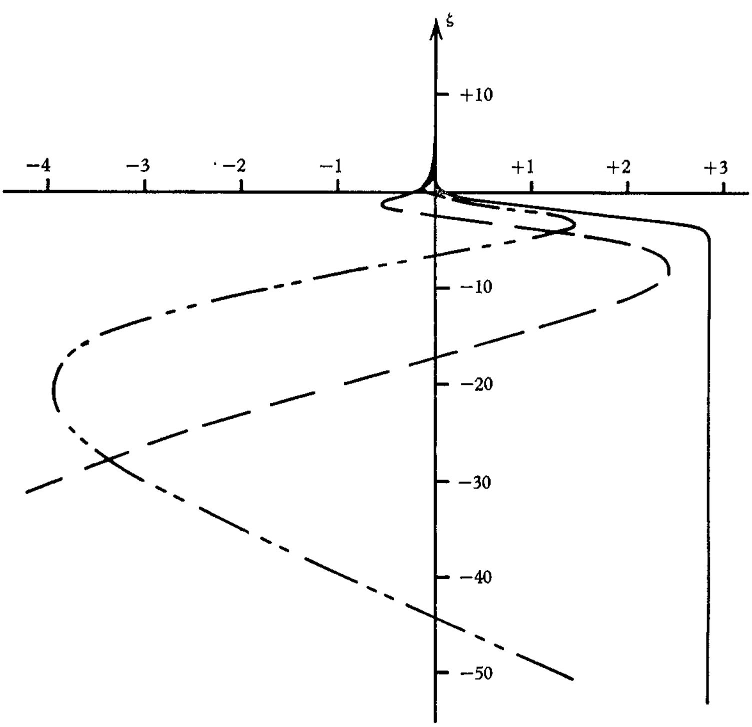
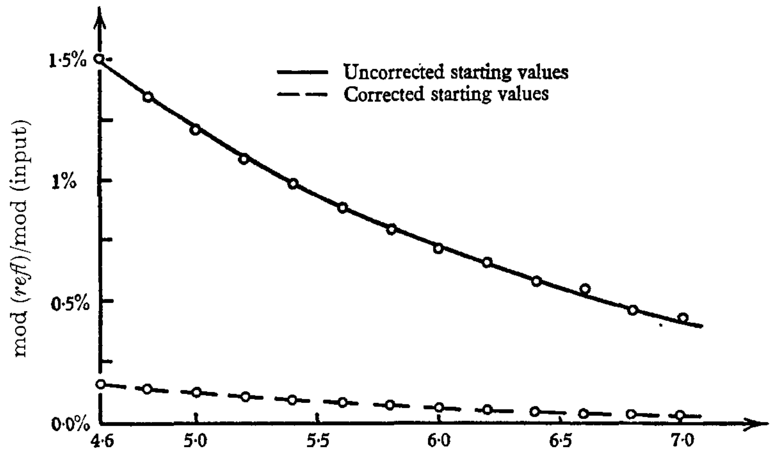


FIGURE 2. ——, Reynolds stress; – – –, imaginary part; —— ——, real part.

wave (see equation (3.1)) a complex coefficient of unity for the transmitted wave would have indicated complete agreement. Figure 1 is a table showing typical coefficients and other relevant results for various values of the parameters. The complete solution through the critical level is shown in figure 2; the magnitude of the viscous terms is completely negligible at the starting-points, and the solution has been matched to a purely inviscid solution there.



Starting-point

FIGURE 3

(*b*) *Reflected wave*

From the numerical calculations a very small reflected wave is predicted. However, careful checking indicates that this is a result of the numerical method, introduced by starting the integration at a finite point, and by using only an approximation to the asymptotic forms of the wave solutions for starting values. When only the first terms of (2.1) and (2.2) are used to give the starting values, the ratio of the modulus of the reflected wave to the modulus of the incident wave decreases approximately as S^{-3} with increasing starting-point S. Using the first two terms reduces this ratio by a factor of 10, and it now decreases as S^{-6}. S^{-3} and S^{-6} are, of course, the order of the first neglected term in the starting values in each case. Figure 3 is a graph of the ratio of the moduli of the reflected and incident waves for different starting points. Extrapolation to infinite starting point implies no reflected wave.

(*c*) *Reynolds stress*

The Reynolds stress for a typical solution is shown in figure 2. It is constant, as expected, throughout the regions where viscosity plays no part, but decreases almost linearly through a region *just below* the critical level to the small value it has above the level. The ratio of the stress at $-\infty$ to that at $+\infty$ is found to be close to the value of $-e^{2\mu\pi}$ predicted by Booker & Bretherton (see figure 1). The fact that the decrease in stress takes place below the critical level indicates, by (1.8), that the momentum transfer to the mean flow from the waves, as they are attenuated, takes place in this region.

(*d*) *Critical layer thickness*

From the numerical results obtained, it appears that the effect of viscosity is important only in a critical layer around the critical level, a layer thicker below the critical level than above it (for upward travelling waves). In the particular cases investigated, the total thickness was about seven units of ξ, i.e. about $7z_0$ true units of height, $5z_0$ of this lying below the critical level. For a horizontal wavelength of 10 km in the atmosphere, z_0 is about 1 m, while for a wavelength of 10 cm in water in the laboratory, z_0 is about 0·2 cm. The figure of 1 m for the atmosphere is negligibly thin, indicating that in practice turbulent dissipation will take over from viscosity.

5. Conclusion

From the numerical calculations performed on the full sixth-order equation, the matching condition for a gravity wave passing through a critical level is found to be independent of viscosity and heat conduction, although the actual form of the wave near this level is radically altered. No reflected wave occurs and the surplus momentum from a wave which is attenuated is taken up by the mean flow in a thin layer just below the critical level.

The above conclusions, taken in conjunction with those of Booker & Bretherton, and those of Jones (1967), who has investigated the initial value problem in the rotating case, and found the same transmission coefficient yet again, indicate that the actual structure of the wave in the critical layer is a secondary effect, and quite irrelevant to the absorption mechanism, which plays a role analogous to that of viscosity in turbulent dissipation.

I am greatly indebted to Dr. F. P. Bretherton for a lot of very helpful hints and advice in connexion with this paper, and for interesting me in the problem in the first place.

Appendix

By F. P. Bretherton, P. Hazel, S. A. Thorpe† and I. R. Wood‡.

Department of Applied Mathematics and Theoretical Physics, University of Cambridge.

Figures A 1 and A 2 demonstrate critical layer absorption taking place in the laboratory in the lee-wave train behind a small obstacle.

They were obtained using an enclosed rectangular Perspex-sided tank 4 m long, 7·5 cm wide, 10 cm high, which was pivoted about a horizontal axis parallel to the shortest side. It was tilted until the longest side was inclined at about 20° to the horizontal and filled, following Oster (1965)§, with a stratified salt solution (2‰ to 5‰) marked at intervals with layers of dye, and then slowly lowered

† Now at: National Institute of Oceanography, Wormley, Surrey.

‡ Permanent address: University of New South Wales, Sydney, Australia.

§ See p. 74, 'instant' density gradient.

to the horizontal position. After the fluid had come completely to rest, the density was to a good approximation a linear function of the height z, (i.e. N^2 = constant) except very near the upper and lower boundaries, with the dye in predetermined sheets. The tank was then tilted through about 5° for a few seconds and restored to the horizontal. In this time the buoyancy forces set up a shear flow with a velocity profile determined by the stratification and angle and duration of tilt. This flow was independent of distance along the tank (except near the ends), and also independent of time until internal waves had propagated from the ends to the point under consideration. When the latter arrived (after 25 s) the flow

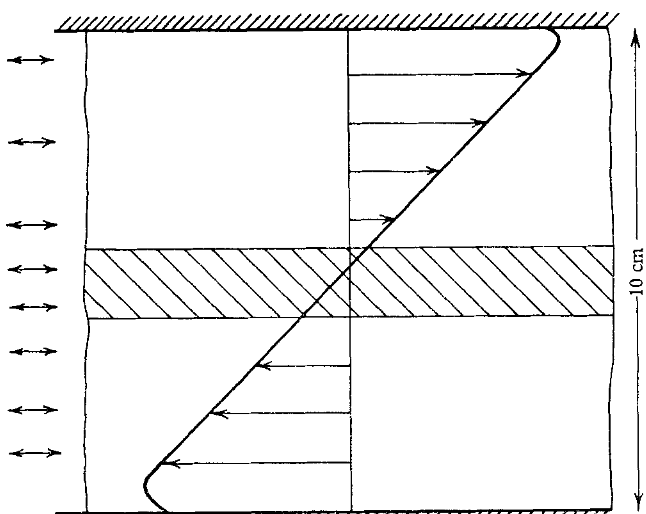


FIGURE A 3. A typical upstream velocity profile is shown, together with the initial levels of the dye streaks, ↔, and the approximate thickness of the critical layer ($7z_0$), as predicted by the numerical results in the main paper, ▧.

reversed. The lee-wave train behind the obstacle (a machined block of Perspex extending across the tank, with triangular cross-section, 4 cm broad at the base, 0·6 cm high) had developed to an apparently steady state before this occurred.

A typical velocity profile (upstream in the lower half of the flow) is shown in figure A 3. The shape of this was obtained from photographs of the distortion of a vertical dye streak. The Richardson number is computed from the known density gradient and the inferred velocity gradient over the central range of heights:

$$R = \frac{1}{N^2\tau^2\sin^2\alpha},$$

where τ = time of tilt, α = angle of tilt. The thickness of the viscous boundary layers on the upper and lower surfaces can be clearly estimated. This increased with time to a maximum of about 1 cm just before flow reversal. For figure A 2, plate 2, the velocity profile was very similar, only the Richardson number was different.

The photographs confirm the following predictions of theory (see also Booker & Bretherton 1967).

(i) If $R > \frac{1}{4}$, the lee waves are confined entirely to the region below the critical level downstream of the obstacle. For the Richardson numbers in the experiments (0·5–200) the amplitude of the wave transmitted to the upper half of the fluid is so small as not to be detectable.

(ii) The amplitude decays rapidly downstream, indicating absorption of wave energy.

(iii) The wave crests tilt forward with increasing z, indicating an upward component of group velocity.

(iv) The wave amplitude increases with z to a maximum just below the viscous layer where absorption is taking place (particularly marked in figure A2). According to theory, in the inviscid region the vertical velocities decrease as the square root of the distance as the critical level is approached, but the vertical *displacements* increase.

It is also worth noting that in figure A2 the wave amplitude is probably too large for linearized theory to be applicable in the critical layer. Although a little fluid just above the critical level is involved in the wave motion, there is still no detectable wave transmission to the upper part of the flow.

Further experiments on this and other problems relating to the stability of stratified shear flow in a rectangular tank are being made by one of the authors (S. A. T.), and the results will be published later. With the present apparatus it proved impossible to achieve a steady wave pattern with Richardson number less than $\frac{1}{4}$. This would, of course, provide a critical test for the theory.

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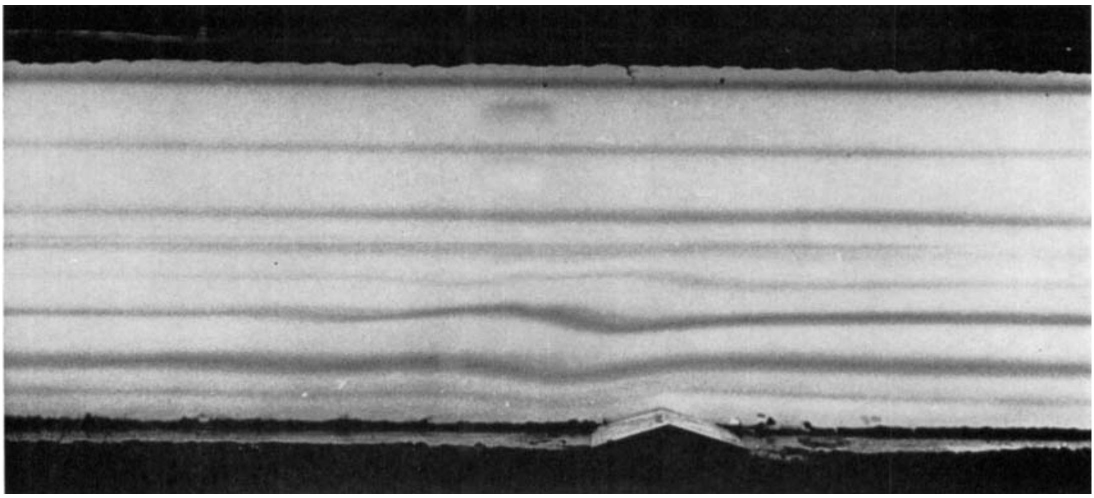

FIGURE A1. $R = 16{\cdot}4$.

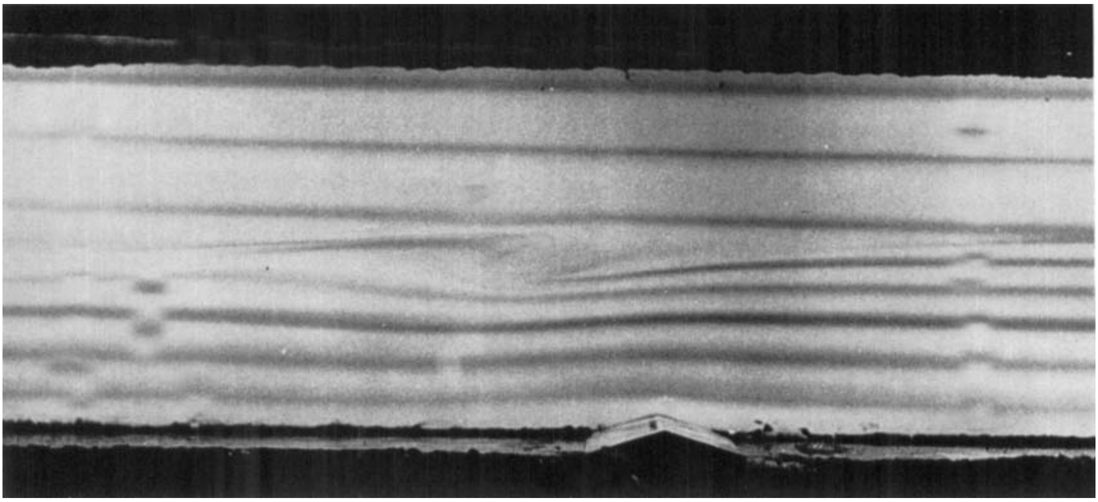

FIGURE A 2. $R = 1{\cdot}2$.